Line Integrals

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MATH 2321

Differentials of Oriented Curves

Let C be an *oriented* curve (in $\mathbb{R}^2$ or $\mathbb{R}^3$), *parametrized* by

$$\mathbf{r}(t),\ \ a \le t \le b\ \ (\text{with } \mathbf{r}'(t) \neq \mathbf{0}).$$

We define the following *differentials* associated to $C/\mathbf{r}$.

- $ds = |\mathbf{r}'(t)|\,dt = |d\mathbf{r}| = \sqrt{dx^2 + dy^2\,(+dz^2)}$
- $dx = x'(t)\,dt$, $dy = y'(t)\,dt$ (and $dz = z'(t)\,dt$)
- $d\mathbf{r} = \mathbf{r}'(t)\,dt = \mathbf{i}\,dx + \mathbf{j}\,dy\ \ (+\mathbf{k}\,dz)$

These are used to integrate functions and vector fields along C.

Example 1

Example

Evaluate $\int_C 4x^3\,ds$*, where* C *is the portion of* $y = x^3 - 1$ *in the fourth quadrant, oriented upward.*

Since C is the graph of a function, we set $x = t$ and $y = t^3 - 1$. We see from the graph that $0 \le t \le 1$. Now

$$dx = dt, \;\; dy = 3t^2\,dt \;\; \Rightarrow \;\; ds = \sqrt{1^2 + (3t^2)^2}\,dt = \sqrt{1 + 9t^4}\,dt.$$

Hence

$$\int_C 4x^3\,ds = \int_0^1 4t^3\sqrt{1+9t^4}\,dt = \left.\frac{2(1+9t^4)^{3/2}}{27}\right|_0^1 = \boxed{\frac{-2+20\sqrt{10}}{27}}.$$

Example 2

Example

Evaluate $\int_C xy^2\,dx$, $\int_C xy^2\,dy$ *and* $\int_C xy^2\,ds$ *where* C *is the right half of* $x^2+y^2=16$, *oriented counterclockwise.*

Since C is part of a circle at the origin, we set $x = 4\cos t$ and $y = 4\sin t$. We see from the graph that $-\pi/2 \le t \le \pi/2$. Now

$$dx = -4\sin t\,dt, \;\; dy = 4\cos t\,dt \quad \Rightarrow \quad ds = 4\,dt.$$

Hence

$$\int_C xy^2\,ds = \int_{-\pi/2}^{\pi/2} 4^4 \cos t\,\sin^2 t\,dt = 4^4 \cdot \left.\frac{\sin^3 t}{3}\right|_{-\pi/2}^{\pi/2} = \boxed{\frac{2^9}{3}}.$$

Example 2
(Continued)

Furthermore

$$\int_C xy^2\,dx = -4^4\int_{-\pi/2}^{\pi/2}\cos t\,\sin^3 t\,dt = -4^4\cdot\left.\frac{\sin^4 t}{4}\right|_{-\pi/2}^{\pi/2} = \boxed{0}.$$

And finally

$$\begin{aligned}\int_C xy^2\,dy &= 4^4\int_{-\pi/2}^{\pi/2}\cos^2 t\,\sin^2 t\,dt = 4^4\int_{-\pi/2}^{\pi/2}\left(\frac{\sin 2t}{2}\right)^2 dt\\ &= 4^3\int_{-\pi/2}^{\pi/2}\frac{1-\cos(4t)}{2}\,dt = 2^5\cdot\left.\left(t-\frac{\sin(4t)}{4}\right)\right|_{-\pi/2}^{\pi/2}\\ &= \boxed{2^5\pi.}\end{aligned}$$

Example 3

Example

Evaluate $\int_C \langle x - y, xy\rangle \cdot d\mathbf{r}$, *where* C *is the portion of* $x^2 + y^2 = 4$ *from* $(2, 0)$ *to* $(0, -2)$, *counterclockwise.*

Since C is part of a circle at the origin, we set $x = 2\cos t$ and $y = 2\sin t$. We see from the graph that $0 \le t \le 3\pi/2$. Now

$$d\mathbf{r} = \langle -2\sin t, 2\cos t\rangle\, dt,$$

so that

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= (2\cos t - 2\sin t)(-2\sin t) + (4\cos t\,\sin t)(2\cos t)\,dt\\ &= -4\sin t\cos t + 4\sin^2 t + 8\cos^2 t\,\sin t\,dt.\end{aligned}$$

Example 3
(Continued)

Hence

$$\int_C \langle x-y, xy\rangle \cdot d\mathbf{r} = \int_0^{3\pi/2} -4\sin t\cos t + 4\sin^2 t + 8\cos^2 t\,\sin t\,dt$$
$$= \boxed{\frac{2}{3} + 3\pi}.$$

Remarks:

- The first and last terms in the integral can be simultaneously antidifferentiated with the substitution $u = \cos t$.
- The middle term requires the half angle formula $\sin^2 t = \frac{1-\cos 2t}{2}$.

Remarks

Parametrization Independence and Vanishing Tangent

- Any line integral $\int_C$ depends on C only, *not* on the choice of $\mathbf{r}(t)$ describing it.
- We can actually allow $\mathbf{r}' = \mathbf{0}$ at finitely many points in $[a, b]$.
- One must make sure $\mathbf{r}'$ doesn't change direction and retrace part of C at these points.
- The moral is that $\mathbf{r}' \neq \mathbf{0}$ is not as important as $\mathbf{r}$ being *one-to-one*.

Remarks

Some Basic Relationships

Let C be an oriented curve, starting at A and ending at B.

- $\displaystyle\int_C dx = x(B) - x(A)$ and likewise for dy and dz.
- Let $-C$ denote C with the opposite orientation. Then

$$\int_{-C} f\,dx = -\int_C f\,dx, \quad \int_{-C} f\,ds = \int_C f\,ds,$$

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}.$$

- These can be convenient when parametrizing in a particular direction is troublesome (e.g. a clockwise oriented circle).

Piecewise Line Integrals

If C is made up of smooth pieces $C_1, C_2, C_3, \ldots$, then

$$\int_C f\,dx = \int_{C_1} f\,dx + \int_{C_2} f\,dx + \int_{C_3} f\,dx \cdots,$$

and likewise for all other differentials.

Example

Evaluate $\int_C xy\,dx + (x-y)\,dy$, *where* C *is the piecewise linear path from* $(0,0)$ *to* $(2,0)$ *to* $(3,2)$.

A Piecewise Example

C consists of the line segments C_1 from $(0,0)$ to $(2,0)$, and C_2 from $(2,0)$ to $(3,2)$.

On C_1: $y = 0$ and $dy = 0$. Hence $\int_{C_1} xy\,dx + (x-y)\,dy = \boxed{0}$.

On C_2: $\mathbf{r}(t) = \langle 2,0\rangle + t\langle 1,2\rangle = \langle 2+t, 2t\rangle$ with $0 \le t \le 1$.

Thus $dx = dt$ and $dy = 2\,dt$.

So

$$\int_{C_2} xy\,dx + (x-y)\,dy = \int_0^1 (2+t)(2t) + (2+t-2t)2\,dt$$
$$= \int_0^1 2t^2 + 2t + 4\,dt = \frac{2}{3}t^3 + t^2 + 4t\Big|_0^1 = \boxed{\frac{17}{3}}.$$

Interpreting Differentials

Let f be a function and $\mathbf{F}$ be a vector field. The line integrals

$$\int_C f\,ds, \quad \int_C f\,dx \;\; (\text{or } dy \text{ or } dz), \quad \int_C \mathbf{F}\cdot d\mathbf{r}$$

can all be interpreted using Riemann sums.

- ds represents an infinitesimal unit of *arclength* on C.
- dx represents an infinitesimal change in x along C. Likewise with dy and dz.
- $d\mathbf{r}$ represents an infinitesimal displacement along C.

In certain situations these allow us to interpret the line integrals themselves.

Arclength Integrals

If $f = f(x, y)$, then

$$\int_C f\, ds$$

is the (signed) area of the surface between C and the graph of f (a "fence").

In particular, if $f \equiv 1$, then

$$\int_C ds = \text{ arclength of } C.$$

Example 4

Example

Find the arclength of the portion of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ *between* $(1,0,0)$ *and* $(1,0,2\pi)$.

We have

$$d\mathbf{r} = \langle -\sin t, \cos t, 1\rangle\, dt \quad \Rightarrow \quad ds = |d\mathbf{r}| = \sqrt{2}\, dt.$$

So the length is

$$\int_C ds = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\pi\sqrt{2}.}$$

Remark: This can also be computed geometrically by "unrolling" the cylinder that the helix sits on.

Line Integrals of Vector Fields

The differentials dx, dy, dz and $d\mathbf{r}$ can be "interpreted" simultaneously.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then on the one hand we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})\cdot(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)\\ &= \int_C P\,dx + Q\,dy + R\,dz.\end{aligned}$$

But we also have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)dt = \int_a^b \frac{\mathbf{F}\cdot\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_C \mathrm{proj}_{\mathbf{T}}\,\mathbf{F}\,ds,$$

where $\mathbf{T}$ is the tangent vector to C at any point.

Integrating $\text{proj}_{\mathbf{T}}\,\mathbf{F}$

What does the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \text{proj}_{\mathbf{T}}\,\mathbf{F}\,ds$$

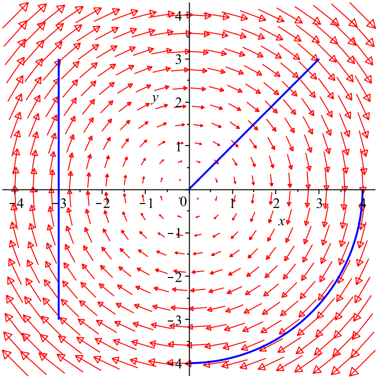
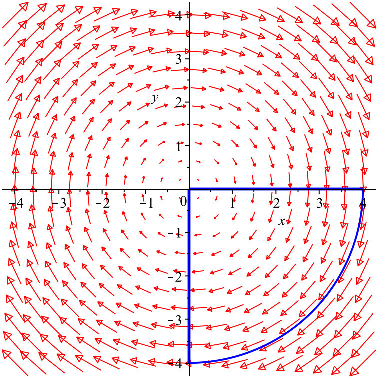
measure?

- Recall that $\text{proj}_{\mathbf{T}}\,\mathbf{F}$ is:
  - positive when $\mathbf{T}$ and $\mathbf{F}$ tend to point in the same direction;
  - zero when $\mathbf{T}$ and $\mathbf{F}$ are orthogonal;
  - negative otherwise.
- The integral therefore measures the extent to which $\mathbf{F}$ "points along" C.

Example 5

Example

For each curve C shown, choose an orientation and determine if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative or zero.

Work

If **F** is a *force field*, we see that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \text{ work done by } \mathbf{F} \text{ on a particle moving along } C.$$

If **F** is the only force acting on the particle, then by Newton's second law, $\mathbf{F} = m\mathbf{r}''$ and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt = \frac{m}{2} \int_a^b \frac{d}{dt}(\mathbf{r}'(t) \cdot \mathbf{r}'(t))\, dt \\ &= \frac{m}{2}(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = \Delta E_k,\end{aligned}$$

the change in kinetic energy from one end of C to the other.

Integrating Conservative Fields

Suppose that **F** is conservative with potential function f.

Then on any curve we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C f_x\,dx + f_y\,dy + f_z\,dz\\ &= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\,dt\\ &= \int_a^b \frac{df}{dt}\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))\\ &= f(\text{end of } C) - f(\text{beg. of } C).\end{aligned}$$

This is the *Fundamental Theorem of Calculus for Line Integrals.*

The Fundamental Theorem

Theorem (Fundamental Theorem of Calculus for Line Integrals)

If $\mathbf{F}$ *is a conservative vector field with potential function* f*, then for any curve* C *from* A *to* B *we have*

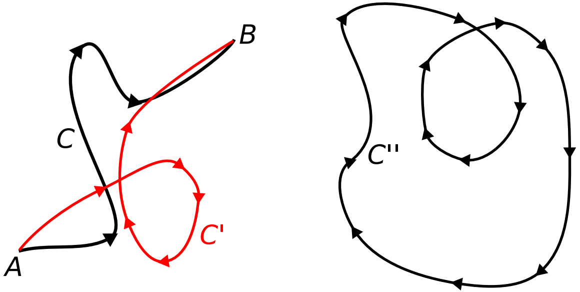
$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Consequently:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is* path independent*; it depends only on the* endpoints *of* C*.*
- $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for any* closed curve C *(when* $A = B$*).*

Path Independence

If $\mathbf{F} = \nabla f$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C''} \mathbf{F} \cdot d\mathbf{r} = 0$.



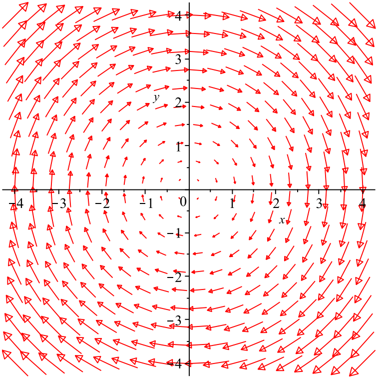
Warning

The FTOC for Line Integrals only *applies when* $\mathbf{F} = \nabla f$. Not all $\mathbf{F}$ *have this property!*

Example 6

Example

Explain why the vector field shown below is not *conservative.*



Example 7

Example

Evaluate $\int_C (x + 2xy)\, dx + x^2\, dy$ *where* C *is the curve consisting of the line segments from* $(0,0)$ *to* $(2,1)$ *to* $(4,3)$ *to* $(5,0)$.

Notice that if $f = x^2 y + \frac{x^2}{2}$, then

$$\nabla f = \langle 2xy + x, x^2 \rangle,$$

which is the vector field under consideration.

So by the Fundamental Theorem

$$\int_C (x + 2xy)\, dx + x^2\, dy = f(5,0) - f(0,0) = \boxed{\frac{25}{2}}.$$

Example 8

Example

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* $\mathbf{F} = x\mathbf{i} + (y+2)\mathbf{j}$ *and* C *is given by* $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $0 \le t \le 2\pi$.

We could compute $d\mathbf{r}$, but instead we notice that

$$\nabla \underbrace{\left(\frac{x^2}{2} + \frac{y^2}{2} + 2y \right)}_{f} = x\mathbf{i} + (y+2)\mathbf{j} = \mathbf{F}.$$

Hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(2\pi)) - f(\mathbf{r}(0)) = f(2\pi, 0) - f(0,0) = \boxed{2\pi^2}.$$

Example 9

Example

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$ *and* C *is the line segment from* $(1,0,0)$ *to* $(3,4,2)$.

Not hard to parametrize a line segment, but again we have

$$\nabla \underbrace{(xy + xz + yz)}_{f} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k} = \mathbf{F}.$$

So once more

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(3,4,2) - f(1,0,0) = \boxed{26}.$$

Characterizing Conservative Fields

- Path independence of line integrals turns out characterize conservative vector fields.
- If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in some domain, it is possible to construct a potential function by "integrating $\mathbf{F}$ to X".
- This means the Fundamental Theorem is actually an "if and only if" result.

Theorem

A vector field $\mathbf{F}$ *with domain* Ω *is conservative if and only if* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is path independent for all* $C \subset \Omega$.

Derivatives of Conservative Fields

Testing path independence for *all* curves can be difficult. How else might we identify a conservative field?

Notice that

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0},$$

by Clairaut's theorem.

Moral

If a field $\mathbf{F}$ *is conservative, then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Example 10
A Nonconservative Field

The vector field $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ is *not* conservative since

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$= (-y)\,\mathbf{i} - (z)\,\mathbf{j} + (-x)\,\mathbf{k}$$

$$= -(y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k})$$

$$\neq \mathbf{0}.$$

Closed and Exact Forms

Consider a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. The associated quantity

$$\omega = P\,dx + Q\,dy + R\,dz$$

is called a *(differential) 1-form*.

Notice that we can write

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_C \omega.$$

ω is called *exact* when $\mathbf{F} = \nabla f$. We write $\omega = df$

ω is called *closed* when $\nabla \times \mathbf{F} = \mathbf{0}$. We write $d\omega = 0$.

Exact $\Rightarrow$ Closed, but...

We have seen that:

Theorem

Every exact form is closed.

However:

Warning

Not every closed form is exact! Depending on the domain, there may exist $\mathbf{F}$ *that are not conservative, but for which* $\nabla \times \mathbf{F} = \mathbf{0}$.

That being said, as a consequence of Green's and Stokes' theorems:

Theorem

If Ω *is a* simply connected *domain (e.g.* $\Omega = \mathbb{R}^2$ *or* $\Omega = \mathbb{R}^3$*), then every closed form on* Ω *is exact.*

Example 11

Example

Is $\omega = y(z+2)\,dx + (xz+2x+4)\,dy + (xy+3)\,dz$ *exact? If so, find a potential function.*

Because ω is defined on $\mathbb{R}^3$, and $d\omega = 0$ (exercise),

$$\boxed{\omega \text{ is exact.}}$$

To find a potential we write

$$\begin{aligned} f_x &= y(z+2), \\ f_y &= xz+2x+4, \\ f_z &= xy+3, \end{aligned}$$

then repeatedly integrate and substitute.

Example 11
Continued

We have

$$f = \int f_x \, \partial x = \int y(z+2) \, \partial x = xy(z+2) + \underbrace{g(y,z)}_{\text{“constant”}} .$$

Now compare to the second and third equations:

$$xz + 2x + 4 = f_y = x(z+2) + g_y \;\Rightarrow\; g_y = 4,$$
$$xy + 3 = f_z = xy + g_z \;\Rightarrow\; g_z = 3.$$

Integrate $g_y = 4$:

$$g = \int g_y \, \partial y = \int 4 \partial y = 4y + \underbrace{h(z)}_{\text{“constant”}} .$$

And compare to g_z:

$$3 = g_z = h'(z) \;\Rightarrow\; h(z) = 3z + C.$$

Example 11

Continued, continued

So the potentials of ω are

$$f = xy(z+2)+g = xy(z+2)+4y+h = \boxed{xy(z+2)+4y+3z+C}.$$

Remarks:

- The constant C is arbitrary. Any choice will satisfy $\omega = df$.
- If ω is known to be exact, the process above will *always* find f.
- In simple cases one can also find f by "guess and check."

Example 12

A Non-Exact Closed Form

Consider the 1-form $\omega = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ on the domain $\mathbb{R} - \{\mathbf{0}\}$, which is *not* simply connected.

Let C be the unit circle, traversed counterclockwise, which can be parametrized by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ \ 0 \le t \le 2\pi.$$

Thus $dx = -\sin t\,dt$, $dy = \cos t\,dt$ and $x^2 + y^2 = 1$, so that

$$\int_C \omega = \int_0^{2\pi} (-\sin t)(-\sin t) + (\cos t)(\cos t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

So ω is *not* exact.

However

$$d\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 2x^2 - 2y^2}{(x^2+y^2)^2} = 0,$$

so that ω *is* closed!

(More) Work

If $\mathbf{F} = \nabla f$ is a *conservative force field*, then $E_p = -f$ is the *potential energy*. If C is any path from A to B,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = E_p(A) - E_p(B).$$

If a particle moves along C under the influence of $\mathbf{F}$ only, by earlier work

$$E_k(B) - E_k(A) = \int_C \mathbf{F} \cdot d\mathbf{r} = E_p(A) - E_p(B),$$

or

$$E_p(A) + E_k(A) = E_p(B) + E_k(B).$$

That is, the *total energy* $E = E_p + E_k$ is *conserved* by $\mathbf{F}$.

Conservative Fields are ... Conservative!

This shows:

Theorem

A conservative force field obeys the Law of Conservation of Energy.

Put another way:

Moral

The quantity "conserved" by a conservative vector field is the total energy.